MATH4210: Financial Mathematics Tutorial 7

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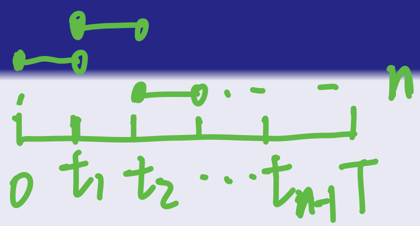
Content Review

Question

- *Let θ_t be a simple process.*



(t_0)

$$\int_0^T \theta_t dB_t = \sum_{i=0}^{n-1} \alpha_i (B_{t_{i+1}} - B_{t_i}) = \alpha_0 (B_{t_1} - 0) + \cdots + \alpha_{n-1} (B_{t_n} - B_{t_{n-1}}).$$

- *For each $\theta_t \in \mathbb{H}^2([0, T])$, there exists a sequence of simple processes $(\theta_t^n)_{n \geq 1}$ such that*

$$\lim_{n \to \infty} \mathbb{E}\Big[\int_0^T (\theta_t - \theta_t^n)^2 dt \Big] = 0.$$

Content Review

Question

- *If $(\theta_t^n)_{n\geq 1}$ is a sequence of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T (\theta_t - \theta_t^n)^2 dt\Big] = 0,$$

then the sequence of stochastic integrals $\int_0^T \theta_t^n dB_t$ has a unique limit in $\mathbb{L}^2(\Omega)$ as $n \to \infty$. And we define

$$\int_0^T \theta_t dB_t := \lim_{n\to\infty} \int_0^T \theta_t^n dB_t$$

as this limitation.

Content Review

Question

- *Let* $\theta_t \in \mathbb{H}^2([0,T])$, *then*

$$\mathbb{E}\Big[\int_0^T \theta_t dB_t\Big] = 0,\ \mathbb{E}\Big[\Big(\int_0^T \theta_t dB_t\Big)^2\Big] = \mathbb{E}\Big[\int_0^T \theta_t^2 dt\Big].$$

- *Let* H_t *be a adapted process and* $\Delta B^n_{k+1} := B_{t^n_{k+1}} - B_{t^n_k}$, *then* $\sum_{k=0}^{n-1} \cancel{H_{t^n_k}}((\Delta B^n_{k+1})^2 - \Delta t) \to 0$, *in probability.*

$\Delta t = \frac{T}{n}$

$0\ (t_0)\quad t_1\quad t_2\ \cdots\cdots\ T$

$$\sum_{k=0}^{n-1}((\Delta B^n_{k+1})^2 - \Delta t) \to 0$$

$$\sum_{k=0}^{n-1}(\Delta B^n_{k+1})^2 - \underbrace{\sum_{k=0}^{n-1}\Delta t}_{\frac{T}{n}\cdot n} \to 0$$

$$\Rightarrow \sum_{k=0}^{n-1}(\Delta B^n_{k+1})^2 \to T \text{ in probability}$$

Stochastic Integration

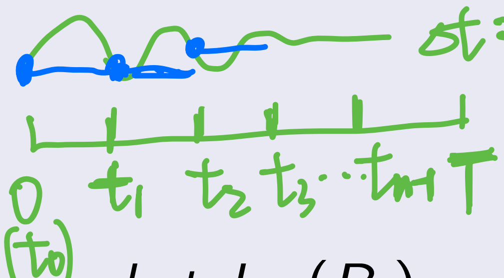
$\overset{(n_j)}{\Rightarrow} \sum_{k=0}^{n_j-1} \left(\Delta B^{n_j}_{k+1}\right)^2 \to T$, a.s

$X_n \to X$ in prop.

$\Rightarrow$ subsequence $X_{n_j} \to X$ a.s.

Question

For fixed $T > 0$, prove that



$\Delta t = \frac{T}{n}$

$$\int_0^T B_t dBt = \frac{1}{2}B_T^2 - \frac{1}{2}T$$

from sketch. $(B_t)_{t\geq 0}$ is a standard Brownian motion.

$\theta^n_t := B_{t_k}$, $t \in [t_k, t_{k+1})$

$B_t \approx \theta_t$

Stochastic Integration

① $\lim_{n\to\infty} E\left[\int_0^T (B_t-\theta_t^n)^2 dt\right]=0$

$E\left[\int_0^T (B_t-\theta_t^n)^2 dt\right]$

$= E\left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (B_t-\theta_t^n)^2 dt\right]$

$= E\left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (B_t-B_{t_k})^2 dt\right]$

$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} E(\underbrace{B_t-B_{t_k}}_{\sim N(0,\,t-t_k)})^2 dt$

$\left(Var(X) = E(X^2) - \underbrace{[E(X)]^2}_{=0} \right)$

$< \frac{T}{n}$ $= t-t_k$

$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (t-t_k)\, dt$

$< \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \frac{T}{n}\, dt = \sum_{k=0}^{n-1} \frac{T}{n}\,\underbrace{(t_{k+1}-t_k)}_{\Delta t = \frac{T}{n}}$

② $\int_0^T B_t\, dB_t = \lim_{n\to\infty}\int_0^T \theta_t^n\, dB_t$

$\int_0^T \theta_t^n\, dB_t \to -\frac{1}{2}T+\frac{1}{2}B_T^2$

$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \theta_t^n\, dB_t$

$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} B_{t_k}\, dB_t$

$= \sum_{k=0}^{n-1} B_{t_k}(B_{t_{k+1}}-B_{t_k})$

$= \sum_{k=0}^{n-1} -\frac{1}{2}\left(-2B_{t_k}B_{t_{k+1}}+2B_{t_k}^2\right)$

$= \sum_{k=0}^{n-1} -\frac{1}{2}\left[(B_{t_k}-B_{t_{k+1}})^2 - B_{t_{k+1}}^2 + B_{t_k}^2\right]$

$= \sum_{k=0}^{n-1} -\frac{1}{2}\left[\underbrace{(B_{t_k}-B_{t_{k+1}})^2}_{\Delta B_{t_k}^2}\right] - \sum_{k=0}^{n-1} \frac{1}{2}\left[B_{t_{k+1}}^2 - B_{t_k}^2\right]$

$\left(\sum_{k=0}^{n-1} \Delta B_{t_k}^2 \to T,\ a.s\right)$

$\to -\frac{1}{2}T+\frac{1}{2}B_T^2$

$\sum_{k=0}^{n-1}\left[B_{t_{k+1}}^2 - B_{t_k}^2\right]$

Ito Formula

$= \frac{T^2}{n^2} \cdot n = \frac{T^2}{n} \to 0$ (subsequence, a.s)

$B_{t_1}^2 - B_{t_0}^2 + B_{t_2}^2 - B_{t_1}^2$

$\cdots \quad B_{t_n}^2 - B_{t_{n-1}}^2$

$= B_{t_n}^2 - \underbrace{B_{t_0}^2}_{=0}$

$= B_T^2$

$Y_t = f(B_t)$ $\quad dB_t \sim O(t^{\frac{1}{2}})$ $\quad dt \sim O(t)$

$dY_t = f'(B_t)dB_t + \frac{f''(B_t)}{2}\underbrace{(dB_t)^2}_{\approx dt}$

$= f'(B_t)dB_t + \frac{f''(B_t)}{2}dt$

Question

Consider a standard Brownian motion $(B_t)_{t \geq 0}$. *Let* $T > 0$, *compute*

(a) $\int_0^T B_t dBt$ $\quad \int_0^T d\left(\frac{B_t^2}{2}\right)$

(b) $\int_0^T \boxed{\exp(B_t - \frac{1}{2}t)}dt$

~~*using Ito formula.*~~

(a) $f(x) := x^2 \quad f' = 2x \quad f'' = 2$

$df(B_t) = f'(B_t)dB_t + \frac{f''(B_t)}{2}dt$

$dB_t^2 = 2B_t dB_t + dt$

$\xrightarrow{0 \to T} \int \quad B_T^2 - \underbrace{B_0^2} = 2 \cdot \int_0^T B_t dB_t + T - 0$

$\Rightarrow B_T^2 = 2\int_0^T B_t dB_t + T$

$\Rightarrow \int_0^T B_t dB_t = \frac{B_T^2}{2} - \frac{T}{2}$

Stochastic Integration

$f(t, B_t) := Y_t$

$$dY_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \frac{\partial^2 f}{\partial^2 x} \cdot \frac{1}{2} \underbrace{(dB_t)^2}_{\approx dt} \qquad \boxed{\cancel{dB_t dt \approx O(t^{1.5})}}$$

$$= \left(\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial^2 x} \cdot \frac{1}{2} \right) dt + \frac{\partial f}{\partial x} dB_t$$

$$\underline{f(t, x) = e^{x - \frac{1}{2}t}} \qquad \frac{\partial f}{\partial t} = e^{x-\frac{1}{2}t} \cdot \left(-\frac{1}{2}\right) \cdot \quad \frac{\partial f}{\partial x} = e^{x-\frac{1}{2}t} \cdot \quad \frac{\partial^2 f}{\partial x^2} = e^{x-\frac{1}{2}t}$$

$$df(t, B_t) = \underbrace{\left(\left(-\frac{1}{2}\right) e^{B_t - \frac{1}{2}t} + e^{B_t - \frac{1}{2}t} \cdot \frac{1}{2} \right)}_{=0} dt + e^{B_t - \frac{1}{2}t} dB_t$$

$$\overset{\int_0^T}{\Longrightarrow} f(T, B_T) - f(0, B_0) = \int_0^T e^{B_t - \frac{1}{2}t} dB_t$$

$$\underline{e^{B_T - \frac{1}{2}T} - e^0} = \int_0^T e^{B_t - \frac{1}{2}t} dB_t$$

$$\underline{e^{B_T - \frac{1}{2}T} - 1} = \int_0^T e^{B_t - \frac{1}{2}t} dB_t$$